Near-horizon circular orbits and extremal limit for dirty rotating black holes

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We consider generic rotating axially symmetric "dirty" (surrounded by matter) black holes. Near-horizon circular equatorial orbits are examined in two different cases of near-extremal (small surface gravity κ) and exactly extremal black holes. This has a number of qualitative distinctions. In the first case, it is shown that such orbits can lie as close to the horizon as one wishes on suitably chosen slices of space-time when $\kappa \to 0$. This generalizes observation of T. Jacobson Class. Quantum Grav. 28 187001 (2011) made for the Kerr metric. If a black hole is extremal ($\kappa = 0$), circular on-horizon orbits are impossible for massive particles but, in general, are possible in its vicinity. The corresponding black hole parameters determine also the rate with which a fine-tuned particle on the noncircular near-horizon orbit asymptotically approaches the horizon. Properties of orbits under discussion are also related to the Bañados-Silk-West effect of high energy collisions near black holes. Impossibility of the on-horizon orbits in question is manifestation of kinematic censorship that forbids infinite energies in collisions.

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I.   INTRODUCTION

Circular orbits near rotating black holes is an important issue in astrophysics and are of interest theoretically on their own since they possess a number of nontrivial properties. There are three main types of such orbits - the photon orbit, the marginally bound and the marginally stable ones. The latter is also called the innermost stable circular orbit (ISCO). It was shown in a "classic" paper [1] for the Kerr metric that in the extremal limit, all three orbits share the same value of the Boyer-Lindquist radial coordinate r_0 that coincides with the horizon radius r_H. Hereafter, subscript "H" refers to the quantity taken on the horizon. However, it was stressed in [1] that it does not mean that they coincide with the horizon. The radial coordinate becomes degenerate there and is unsuitable for analysis. A more thorough inspection showed that the proper distance between each orbit and the horizon and between different orbits is not zero in the limit under discussion (moreover, the distance between the marginally bound orbit and ISCO even diverges).

Meanwhile, some subtleties concerning the properties of these orbits remained unnoticed until recently. As was pointed out in [2], the very meaning of location of orbits depends crucially on what slice of space-time is chosen for inspection. There are such slices that the proper distance to the horizon tends to zero, so in the extremal limit these orbits asymptotically approach the horizon.

Formally, $\lim_{\kappa \to 0} r_0 = r_H$, where r_0 is the radius of the orbit, κ is the surface gravity ($\kappa = 0$ for extremal black holes). However, the situation when a black hole is extremal from the very beginning should be considered separately. Time to time debates on this issue have been continuing [3], [4] in spite of the fact that, clearly, a trajectory of a massive particle cannot lie on the light-like surface. Detailed analysis of this circumstance was done in [5] for the Kerr metric. With the help of a coordinate system suggested below, we carry out analysis for a generic axially symmetric rotating black hole and show that the corresponding circular orbit is indeed fake.

Although such orbits with $r_0 = r_H$ are fake, we show that the values of black hole parameters for which the radii of these orbits coincide with r_H do have some physical meaning. This meaning is twofold. Circular orbits with radii slightly above the horizon do exist, then in the main approximation the black hole parameters under discussion are not arbitrary but take some fixed values. On the other hand, these values also characterize dynamics, when a

particle moves along the trajectory that is not exactly circular. Then, the rate with which a particle asymptocially approaches the horizon changes when black hole angular momentum (or another parameter) crosses these values in the space of parameters.

The aim of the present work is to give a general description of the equatorial near-horizon orbits in question for a generic stationary axially-symmetric black hole. We do not specify its metric and do not assume it to be necessarily the Kerr or Kerr–Newman one. This deviation can be attributed to matter that inevitably surrounds a black hole in astrophysics. In this sense, a black hole is "dirty". In the equatorial plane, we introduce a coordinate system that generalizes the version of the coordinate system suggested for the Kerr metric in [6]. Hopefully, this system can be of interest not only for the issue of near-horizon circular orbits but can have some general value. We consider separately the extremal limits of nonextremal black holes and extremal black holes that leads to qualitatively different results.

The properties of near-horizon circular orbits are intimately related to the so-called Bañados-Silk-West (BSW) effect [8]. According to this effect, if two particles collide near a black hole, the energy in their centre of mass $E_{c.m.}$ can be, under certain conditions, formally unbound. The fact that the on-horizon orbit of a massive particle is fake turns out to be intimately connected with impossibility to gain infinite $E_{c.m.}$. This quantity remains finite in any act of collision, although it can become as large as one likes (this can be called "the kinematic censorship"). The relation between the high energy collisions and properties of ISCO close to the horizon was considered first in [5] for the near-extremal Kerr metric and generalized in [12] for dirty black holes. In the present paper, we discuss the aforementioned relation both for near-extremal and extremal black holes. We also give geometric explanation to the high energy collisions of particles near ISCO. This can be considered as a counterpart of geometric explanation given for the standard BSW effect [7].

We would like to stress that studies of motion of particles in the background of the Kerr-Newman black hole has attracted interest during many years until recently [9] - [11]. In general, details of motion are quite complicated and depend crucially on the concrete properties of the metric. In our approach, we restrict ourselves by the near-horizon region only. This enables us to trace some features that rely on the properties of the horizon in a model-independent way. This can be thought of as one more manifestation of universality of black hole physics.

Throughout the paper, we put fundamental constants $G = c = 1$.

II. COORDINATE SYSTEM AND EQUATIONS OF MOTION

We consider the metric of the form

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_\theta d\theta^2, \tag{1}$$

where the metric coefficients do not depend on t and ϕ. Correspondingly, the energy $E = -mu_0$ and angular momentum $L = mu_\phi$ are conserved. Here, m is the particle's mass, u^μ is the four-velocity. In what follows, we restrict ourselves by motion in the equatorial plane $\theta = \frac{\pi}{2}$ only. Then, it is convenient to redefine the radial coordinate $r \to \rho$ in such a way that

$$ds^2 = -N^2 dt^2 + g_\phi (d\phi - \omega dt)^2 + \frac{d\rho^2}{N^2}. \tag{2}$$

Equations of motion for a test particles moving along the geodesic read

$$m\frac{dt}{d\tau} = \frac{X}{N^2}, \tag{3}$$

$$X \equiv E - L\omega, \tag{4}$$

$$m\frac{d\phi}{d\tau} = \frac{L}{g_\phi} + \frac{\omega X}{N^2}, \tag{5}$$

where τ is the proper time.

The forward in time condition $\frac{dt}{d\tau} > 0$ requires

$$X \geq 0, \tag{6}$$

where $X = 0$ is possible on the horizon only, where $N = 0$.

Using also the normalization condition $u_\mu u^\mu = -1$, one can infer that

$$m^2 (\frac{d\rho}{d\tau})^2 = Z^2 \equiv X^2 - N^2 (\frac{L^2}{g} + m^2). \tag{7}$$

We also may introduce new (barred) coordinates according to

$$dt = d\bar{t} - \frac{\chi d\rho}{N^2}, \tag{8}$$

$$d\phi = d\bar{\phi} - \frac{w d\rho}{N^2}, \rho = \bar{\rho}, \tag{9}$$

where the functions χ and w depend on ρ only. Then,

$$d\phi - \omega dt = d\bar{\phi} - \omega d\bar{t} + \frac{\omega\chi - w}{N^2} d\rho, \tag{10}$$

$$g_{\rho\rho} = \frac{1 - \chi^2}{N^2} + \frac{g_\phi(\omega\chi - w)^2}{N^4}. \tag{11}$$

We choose the functions χ, w to kill divergences in the metric coefficient $g_{\rho\rho}$. To this end, we require

$$\chi^2 = 1 - \mu N^2, \tag{12}$$

$$\omega\chi - w = N^2 h, \tag{13}$$

where $h(\rho)$ and $\mu(\rho)$ are bounded functions, $h(\rho_H) \neq 0$, $\mu(\rho_H) \neq 0$. We obtain

$$d\phi - \omega dt = d\bar{\phi} - \omega d\bar{t} + h d\rho. \tag{14}$$

It is convenient to choose $\mu = 1$, $h = -w$. Then,

$$\chi = \sqrt{1 - N^2}, \; w\chi = \omega. \tag{15}$$

As a result,

$$ds^2 = -N^2 d\bar{t}^2 + g_\phi(d\bar{\phi} - \omega d\bar{t})^2 + (1 + g_\phi w^2) d\rho^2 - 2g_\phi(d\bar{\phi} - \omega d\bar{t}) w d\rho + 2\chi d\bar{t} d\rho. \tag{16}$$

It can be also rewritten in the form

$$ds^2 = -d\bar{t}^2 + (1 + g_\phi w^2)(d\rho + \chi d\bar{t} - \frac{g_\phi w}{1 + g_\phi w^2} d\bar{\phi})^2 + \frac{g_\phi}{1 + g_\phi w^2} d\bar{\phi}^2. \tag{17}$$

For the energy and angular momentum we have

$$\mathcal{E} \equiv \frac{E}{m} = \frac{\bar{E}}{m} = -g_{\bar{t}\bar{t}}\frac{d\bar{t}}{d\tau} + \omega g_\phi \frac{d\bar{\phi}}{d\tau} - (\omega g_\phi w + \chi)\frac{d\rho}{d\tau}, \tag{18}$$

$$\mathcal{L} \equiv \frac{L}{m} = \frac{\bar{L}}{m} = g_\phi \frac{d\bar{\phi}}{d\tau} - \omega g_\phi \frac{d\bar{t}}{d\tau} - g_\phi w \frac{d\rho}{d\tau}. \tag{19}$$

These quantities can be also written as $\mathcal{E} = -u_\mu \xi^\mu$, $\mathcal{L} = u_\mu \eta^\mu$, where ξ^μ and η^μ are Killing vectors responsible for time translation and azimuthal rotation, respectively, $u^\mu = \frac{dx^\mu}{d\lambda}$, λ is the affine parameter along the geodesic. Therefore, the expressions (18), (19) for $\mathcal{E}$ and $\mathcal{L}$ are valid both in the massive and massless cases.

Let us consider motion in the inward direction, so $\frac{d\rho}{d\tau} < 0$. It follows from (7) and (8) that

$$m\frac{d\bar{t}}{d\tau} = \frac{X - \chi Z}{N^2}, \tag{20}$$

$$\frac{d\bar{\phi}}{d\tau} = \frac{L}{mg_\phi} + \frac{\omega X - wZ}{mN^2}. \tag{21}$$

One finds from (20)

$$\bar{t} = -\int \frac{d\rho(X - \chi Z)}{ZN^2}.$$ (22)

If $X_H \neq 0$, near the horizon $Z \approx X_H + O(N^2)$, so the integral converges. Thus, in contrast to the time t (analogue of the Boyer-Lindquist time), the time $\bar{t}$ required to reach the horizon is finite. Only for a special trajectory with $X_H = 0$ (so-called critical particles - see below) $\bar{t}$ is infinite.

In the particular case $L = 0$, $E = m$ we have from (20) and (7), (15) or from (18), (19) that

$$\frac{d\bar{t}}{d\tau} = 1, \frac{d\bar{\phi}}{d\tau} = 0,$$ (23)

so $\bar{\phi} = const$, $\bar{t}$ coincides with the proper time, that agrees with the form of the metric (17). It generalizes the corresponding property of the Kerr metric in Doran coordinates [6], [2].

As now $g_{\rho\rho}$ is bounded near the horizon, the proper distance to the horizon remains finite on the slice $\bar{t} = const$, including the extremal limit.

III.   KERR METRIC

In the particular case of the Kerr metric, we have in the equatorial plane $\theta = \frac{\pi}{2}$

$$\frac{d\rho}{dr} = \sqrt{\frac{r^3}{r^3 + ra^2 + 2Ma^2}},$$ (24)

$$N^2 = \frac{\Delta r^2}{(r^2 + a^2)^2 - a^2\Delta} = \frac{r(r^2 - 2Mr + a^2)}{r^3 + ra^2 + 2Ma^2},$$ (25)

$$g_\phi = r^2 + a^2 + \frac{2M}{r}a^2,$$ (26)

$$\omega = \frac{2Ma}{r^3 + ra^2 + 2Ma^2}.$$ (27)

Here, M is a black hole mass, $a = \frac{J}{M^2}$, J being a black hole angular momentum. The choice

$$w = \frac{a}{r^2 + a^2}\sqrt{\frac{2M(r^2 + a^2)}{r^3 + ra^2 + 2Ma^2}}$$ (28)

satisfies eq. (12), (13). Substituting it into (17), we obtain our metric in the form

$$ds^2 = -d\bar{t}^2 + (\alpha^{-1}\beta dr + \alpha(d\bar{t} - ad\bar{\phi}))^2 + (r^2 + a^2)d\bar{\phi}^2,$$ (29)

$$\beta^2 = \frac{2Mr}{r^2 + a^2}, \alpha^2 = \frac{2M}{r}.$$ (30)

that corresponds to eq. (18) of [6] and eqs. (2), (3) of [2] in which one should put $\theta = \frac{\pi}{2}$.

IV.   CIRCULAR ORBITS FOR NEAR-EXTREMAL BLACK HOLES

Circular orbits ($r = r_0 = const$) are characterized by the conditions

$$Z^2(r_0) = 0, \tag{31}$$

$$\left(\frac{dZ^2}{dr}\right)_{r=r_0} = 0, \tag{32}$$

where Z^2 is defined in (7) and we returned from ρ to r.

For a black hole with arbitrary parameters, the solution of Eqs. (31), (32) is rather complicated even in the simplest case of the Kerr metric. For a generic dirty black hole (1) it is impossible to find the solution analytically at all. However, it turns out that if a nonextremal black hole is close to its extremal state, there are some general features that enabled us to develop a general approach to the analysis of such orbits [12]. If the surface gravity $\kappa = \frac{1}{2}(\frac{\partial N^2}{\partial \rho})_{\rho=\rho_H}$ of a black hole is small ($\kappa = 0$ corresponds to the extremal state), then, it turns out that for ISCO

$$N \sim r_0 - r_H = O(\kappa^{2/3}), \tag{33}$$

and, for the photon and marginally bound near-horizon orbits,

$$N \sim r_0 - r_H = O(\kappa). \tag{34}$$

These relations follow from eqs. (46) and (61) of [12], correspondingly.

Thus in the extremal limit $\kappa \to 0$, the corresponding radius approaches the horizon, $r_0 \to r_H$. However, in doing so, the proper distance within the slice $t = const$ between the horizon and the ISCO

$$l = O(\ln\frac{1}{\kappa}), \tag{35}$$

as is shown in [12]. Between the horizon and the marginally bound or photon orbit

$$l = O(1). \tag{36}$$

These results agree with [1].

If, instead of $t = const$, we choose the slice $\bar{t} = const$, the metric is regular, $g_{\rho\rho}$ is finite in the vicinity of the horizon, so the proper distance

$$\lim_{\kappa \to 0} l = 0 \tag{37}$$

that generalizes observation of Ref. [2].

One reservation is in order. The near-horizon ISCO in the background of near-extremal black holes exist not always but under some additional constraints on the metric parameters. See Ref. [12] and Sec. VII and IX below. In particular, for the near-extremal Reissner-Nordström metric ISCO cannot lie in the vicinity of the horizon (if we restrict ourselves by geodesic trajectories).

V. GEOMETRIC PROPERTIES

The properties of near-horizon obits can be considered from a more general viewpoint. And, this reveals relation between the issue under discussion and another issue - namely, so-called Bañados-Silk-West (BSW) effect (see below). Let us expand the four-velocity of a particle with respect to the basis that contains two lightlike vectors l^μ and N^μ (so $l_\mu l^\mu = 0 = N_\mu N^\mu$) and two spatial vectors. For the case under discussion, when a particle follows a circular equatorial orbit, $r = const$ and also $\theta = const$, so as a matter of fact it is sufficient to use only two basis vectors l^μ and N^μ. We normalize them in such a way that

$$l^\mu N_\mu = -1. \tag{38}$$

Then,

$$u^\mu = \beta N^\mu + \frac{1}{2\alpha} l^\mu. \tag{39}$$

It is convenient to choose

$$l^\mu = (1, 0, 0, \omega + \frac{N}{\sqrt{g_\phi}}), \tag{40}$$

$$N^\mu = \frac{1}{2N^2}(1, 0, 0, \omega - \frac{N}{\sqrt{g_\phi}}), \tag{41}$$

$x^\mu = (t, r, \theta, \phi)$. In the horizon limit,

$$l^\mu \to \xi^\mu + \omega_H \eta^\mu, \tag{42}$$

where ξ^μ and η^μ are the Killing vectors responsible for time translation and rotation, respectively.

Equations of motion for a geodesic particle (3) - (5) tell us that on the circular orbit,

$$X = N\sqrt{\frac{L^2}{g_\phi} + m^2}, \tag{43}$$

$$\beta = \frac{1}{m}(X - \frac{LN}{\sqrt{g_\phi}}), \tag{44}$$

$$\frac{1}{\alpha} = \frac{1}{mN^2}(X + \frac{NL}{\sqrt{g_\phi}}). \tag{45}$$

Thus, for small N, we have from (43) that

$$\beta = pN = O(N), \tag{46}$$

$$\alpha = qN = O(N), \tag{47}$$

where the coefficients are equal to

$$p = \frac{1}{m}(\sqrt{\frac{L^2}{g_\phi} + m^2} - \frac{N}{\sqrt{g_\phi}})_H = O(1), \tag{48}$$

$$q^{-1} = \frac{1}{m}(\sqrt{\frac{L^2}{g_\phi} + m^2} + \frac{N}{\sqrt{g_\phi}})_H = O(1). \tag{49}$$

Also, it follows from (3) - (5) that

$$\frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \omega_H + O(N), \tag{50}$$

$$\frac{l^\phi}{l^t} = \omega_H + O(N). \tag{51}$$

In this sense, the orbit does indeed become asymptotically parallel to the horizon generator, for which $\frac{d\phi}{dt} = \omega_H$. However, although the coefficient at l^μ (which, in turn, approaches the horizon generator) is much larger than that at N^μ, both terms in (39) give comparable contribution to the norm of u^μ [13] because of the property $l_\mu N^\mu = 0$.

VI.   EXTREMAL BLACK HOLE AND FAKE TRAJECTORY ON HORIZON

In the previous section, we discussed the limit $\kappa \to 0$. What happens if $\kappa = 0$ from the very beginning? Formally, one is tempted to put $\kappa = 0$ in (33), (34). However, the process of derivation in [12] essentially implied that although κ is small, $\kappa \neq 0$. For $\kappa = 0$, the asymptotic expansion of the metric coefficient has another form as compared to the

nonextremal case, so it is necessary to proceed anew. According to general properties, for the extremal black holes the expansion of the metric coefficient ω reads [14]

$$\omega = \omega_H - B_1 N + B_2 N^2 + O(N^3), \tag{52}$$

whence one obtains from (4) that

$$X = X_H + (B_1 N - B_2 N^2)L + O(N^3), \tag{53}$$

where B_1 and B_2 are model-dependent coefficients.

If $X_H = 0$, it seems that there exists an exact solution eqs. (31), (32) that reads $N = 0$, $X = 0$. It *would seem* that it describes the trajectory that lies within the horizon. However, it contradicts the fact that the time-like trajectory cannot lie on the null surface.

In Introduction in [4], the attempt was made to assign a meaning to such orbits on the extremal horizon. It is based on the results of [2], where it was revealed that the proper distance to the horizon depends essentially on a slice (see also Sec. II - IV above). Then, an orbit can in a sense lie on the horizon, if the slice $\bar{t} = const$ of the Doran coordinate system [6] is chosen. However, for the situation discussed in [2], black holes are near-extremal but not exactly extremal, there is a parameter κ that can be as small as one likes but nonzero. The proper distance between the circular orbit and the horizon remains finite for any κ on both types of slices ($t = const$ or $\bar{t} = const$). Only asymptotically, in the limit $\kappa \to 0$, the proper distance to the horizon on the slice $\bar{t} = const$ approaches zero, so the claim "the circular orbit lies on the horizon" is to be understood in the asymptotic sense. But now, for extremal black holes, $\kappa = 0$ exactly from the very beginning, so the reference to the approach of [2] does not save the matter. Here, the situation should be considered anew.

Actually, we have in (7) with $\frac{d\rho}{d\tau} = \frac{dr}{d\tau} = 0$ the uncertainty of the type $\frac{0}{0}$. To resolve this uncertainty, the original coordinates are insufficient since they become degenerate on the horizon. Instead, we use the barred coordinates in which the metric takes the form (16).

Below, we generalize approach of Sec.III C of [5], where this issue was considered for the Kerr metric. For the circular orbit, $\frac{d\rho}{d\tau} = 0$. On the horizon, $\bar{N} = 0$, $g_{\bar{t}\bar{t}} = g_\phi \omega_H^2$. After substitution into (18), (19), we see that in this case

$$\mathcal{E} - \omega_H \mathcal{L} = 0. \tag{54}$$

Then, direct calculation gives us

$$u_\mu u^\mu = \mathcal{E}^2 \frac{1}{g_\phi \omega_H^2}. \tag{55}$$

As $u_\mu u^\mu = -1$ for time-like curves and $u_\mu u^\mu = 0$ for lightlike ones, we conclude that for trajectories of physical particles the only possibility is to $\mathcal{E} = 0 = \mathcal{L}$, the curve must be light-like. Thus the time-like on-horizon orbit is fake.

We also infer from (18) and (19) that

$$\frac{d\bar{\phi}}{dt} = \omega_H. \tag{56}$$

Actually, it means that for massless particles, the trajectory in question coincides with the horizon generator.

VII.   CIRCULAR ORBITS FOR NEAR-CRITICAL PARTICLES

In what follows, we use terminology borrowed from works on studies of the BSW effect. We call a particle critical if $X_H = 0$ and usual if $X_H > 0$ is generic. If $X_H \neq 0$ but is small, we call a particle near-critical. It follows from (54) that a particle whose trajectory lies on the horizon is necessarily critical. Such a trajectory can be realized for photon and is forbidden for massive particles. Meanwhile, although a trajectory that lies exactly on the horizon is impossible for massive particles, near-horizon circular orbits are allowed, if corresponding particles are near-critical (not exactly critical), as we will see it below.

It is convenient to use N as a radial coordinate. Then, analogue of eq. (32) reads

$$(\frac{dZ^2}{dN})_{N=N_0} = 0, \tag{57}$$

where N_0 corresponds to the circular orbit. Then, it follows from (31) and (57) that

$$X_0 = N_0 \sqrt{Y_0}, \tag{58}$$

$$\frac{1}{2}(\frac{dZ^2}{dN})_0 = N_0 (B\sqrt{Y} L - Y - \frac{N}{2}\frac{dY}{dN})_0, \tag{59}$$

where

$$Y = \frac{L^2}{g} + m^2, \tag{60}$$

$$B = -\frac{d\omega}{dN}, \tag{61}$$

subscript "0" means that the corresponding quantities are calculated on the circular orbit.

To avoid fake orbits with $N_0 = 0$, we are interested in the solution with $N_0 \neq 0$, so we have

$$(B\sqrt{Y} - Y - \frac{N}{2}\frac{dY}{dN})_0 = 0. \tag{62}$$

Thus there are two equations (58) and (62) for three unknowns X_0, N_0, L. We can fix, say, N_0 and, in principle, calculate X_0 and L_0 (or E and L_0). Eq. (62) is exact.

If we are interested in just near-horizon orbits, we must require $N_0 \approx 0$ and solve the equations iteratively in the form of the Taylor expansion

$$L = L_0 + L_1 N_0 + L_2 N_0^2 + ... \tag{63}$$

In the main approximation in which corrections due to small N_0 are discarded, we obtain

$$B_1 L_0 = \sqrt{Y_H} = \sqrt{\frac{L_0^2}{g_H} + m^2}, \tag{64}$$

where B_1 is the coefficient entering the expansion (52). Also, in the same approximation one can take $X_H \approx 0$ according to (58), whence

$$L_0 = \frac{E}{\omega_H}. \tag{65}$$

It follows from (64), (65) that

$$L_0 = \frac{m}{\sqrt{B_1^2 - \frac{1}{g_H}}}, \; E = \frac{m\omega_H}{\sqrt{B_1^2 - \frac{1}{g_H}}}, \tag{66}$$

where we assume that $m \neq 0$ (for the case $m = 0$, see below). The formulas (66) imply that

$$B_1\sqrt{g_H} > 1. \tag{67}$$

If (64), (67) are violated, there are no circular near-horizon orbits for massive particles. Eq. (64) constraints black hole parameters, for example the angular momentum of a black hole. In particular, for the Kerr-Newman black hole this leads to some distinct values of the black hole angular momentum [4] (see also Sec. VII and IX below). In particular, for the Reissner-Nordström metric $\omega = 0 = B_1$, eq. (67) is violated and this means that the near-horizon ISCO does not exist in this case.

For different types of orbits eq. (64) or (66) leads to different relations.

A.   Photon orbit

Putting $m = 0$ in (64), we obtain

$$B_1^2 = \frac{1}{g_H}. \tag{68}$$

B.   Marginally bound orbit

Putting $E = m$, in the zero approximation we obtain from (31) that

$$L_0 = \frac{m}{\omega_H}. \tag{69}$$

Then, (64) gives us

$$B_1^2 = \frac{1}{g_H} + \omega_H^2. \tag{70}$$

C.   ISCO

By definition, the condition

$$\left(\frac{d^2 Z^2}{dz^2}\right)_0 = 0 \tag{71}$$

should be satisfied for this orbit in addition to (31) and (57) since eq. (71) gives the boundary between stable and unstable orbits. In the main approximation, $\frac{1}{2}\left(\frac{d^2 Z^2}{dz^2}\right)_0 \approx N_0 L^2 (\frac{1}{g^2}\frac{dg}{dN} - 2B_2 B_1)_0$, so we have

$$S \equiv (\frac{1}{g^2}\frac{dg}{dN} - 2B_2 B_1)_H = 0, \tag{72}$$

where we neglected in (72) the difference between the quantities calculated on the circular orbit and on the horizon. In addition to (72), eq. (66) should be satisfied.

D.   Estimate of X_H

It is instructive to find X_H for all types of orbits under discussion. It follows from (53), (58) and (60) that

$$X_H = N_0(\sqrt{Y} - B_1 L) + O(N_0^2). \tag{73}$$

The terms of the order $O(N_0)$ mutually cancel, so

$$X_H = O(N_0^2). \tag{74}$$

Thus the particle that moves on the circular orbit near the extremal black hole turns out to be not only near-critical but should have anomalously small X_0 to keep following such an orbit.

E. Extremal Kerr-Newman black hole

For the equatorial orbit ($\theta = \frac{\pi}{2}$) in the extremal Kerr-Newman background, one can calculate the relevant quantities (using, say, the Boyer-Lindquiste coordinates) and find

$$g_H = \frac{(M^2 + a^2)^2}{M^2}, \tag{75}$$

$$\left(\frac{dg}{dN}\right)_H = 2\frac{(M^2 - a^2)}{M^4}(M^2 + a^2)^2, \tag{76}$$

$$B_1 = \frac{2a}{M^2 + a^2}, \tag{77}$$

$$B_2 = \frac{a^3}{M^2(M^2 + a^2)}, \tag{78}$$

$$\omega_H = \frac{a}{M^2 + a^2}, \tag{79}$$

$$S = \frac{2(M^2 - 2a^2)}{M^2(M^2 + a^2)}. \tag{80}$$

Then, one can find from eqs. (64), (65), (80) that $a = a_0$, where

$$\frac{a_0}{M} = \frac{1}{2}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}} \tag{81}$$

for the photon orbit, marginally bound one and ISCO, respectively. These values agree with [10], [4].

VIII. DYNAMICS OF CRITICAL PARTICLES

In the previous section we saw that for a circular orbit to exist in the immediate vicinity of the horizon, the relations (64), (63) should be satisfied. It *would seem* that, as these relations imply $N_0 = 0$, the corresponding orbit lies exactly on the horizon. However, we already know that such an orbit is fake for massive particles. We may look for the solution for circular orbit in a series with respect to N_0. Then, apart from (63), the similar expansion

should go, say, for the parameter a (if, for definiteness, we consider the Kerr-New-Newman metric):

$$a_c = a_0 + a_1 N_0 + O(N_0^2), \tag{82}$$

where a_0 is the aforementioned value, different for different kinds of orbits, a_c corresponds to the circular orbit.

Let, say, $a_1 < 0$ but a black hole has $a > a_0$. Then, the contradiction with (82) tells us that for $a > a_0$ the circular orbits do not exist near the horizon at all. For example, for the Kerr-Newman metric there is an exact solution describing the circular photon orbit [10] $r_0 = 2M - 2a$ that can be rewritten as

$$a = \frac{M}{2} - \frac{1}{2}(r_0 - M) < \frac{M}{2} \tag{83}$$

for any orbit above the horizon. If $a > \frac{M}{2}$, such a solution ceases to exist.

Formally, then it follows from (59) that the only solution is $N_0 = 0$ (the orbit exactly on the horizon). However, we reject this case since for massive particles it is impossible and for massless ones it is already described above. Thus we are led to the conclusion that, in the absence of a circular orbits, a particle should move. We assume that it moves towards a black hole.

To probe dynamics in this situation, it is instructive to select particles which are exactly critical ($X_H = 0$) since it is this value that was a "candidate" for a circular orbit on the horizon.

It is convenient to expand Z^2 in powers of N:

$$Z^2 = z_2 N^2 + z_3 N^3 + ... \tag{84}$$

Here, the coefficients

$$z_2 = (B_1^2 - \frac{1}{g_H})L^2 - m^2 \tag{85}$$

$$z_3 = [\frac{1}{g_H^2}(\frac{dg}{dN})_H - 2B_1 B_2]L^2, \tag{86}$$

correspond just to (64), (72) but now they are, on general, do not vanish.

It is also convenient to write

$$N = F(\rho)(\rho - \rho_H) = H(r)(r - r_H), \ F(\rho_H) \equiv F_1 \neq 0, \ H(r_H) = H_1 \neq 0. \tag{87}$$

Then, for $z_2 > 0$ we obtain from (7)

$$\frac{dN}{d\tau} \approx -F_1 N \sqrt{z_2}, \tag{88}$$

$$r - r_H \approx r_1 \exp(-F\sqrt{z_2}\tau), \tag{89}$$

where r_1 is another constant.

Let now $z_2 = 0$, $z_3 > 0$. In a similar way, we find from (7) that

$$\frac{1}{F}\frac{dN}{d\tau} \approx -\sqrt{z_3}N^{3/2}, \tag{90}$$

$$r - r_H \approx \frac{4}{F_1^2 \tau H_1^2 z_3}. \tag{91}$$

In general, if $X_H = 0$ and $z_2(r_+) = z_3(r_+) = ... z_k = 0$, $z_{k+1} > 0$,

$$r - r_H \sim \lambda^{-\frac{2}{k-1}}, \tag{92}$$

where λ is the affine parameter (proper time for a massive particle).

IX.   CIRCULAR ORBITS IN VICINITY OF NEAR-EXTREMAL BLACK HOLES

We saw that the circular orbits exist for selected values of black hole parameters only (say, the discrete values of the angular momentum $a = a_0$). In the main approximation, these values do not depend on the posiiton of the near-horizon orbit, in the next approximation a_0 acquires small corrections of the order N_0. This is in sharp contrast with the situation for near-extremal black holes, when the surface gravity κ is small but nonzero. It is shown in [12] that for such black holes circular orbits almost always exist, under rather weak restriction that black hole parameters obey some inequalities (see below). For example, for the case the Kerr-Newman metric it means that M^2 is to be close to $a^2 + Q^2$ (where Q is the black hole electric charge) but the ratio $\frac{a}{M}$ is arbitrary in some finite interval. Thus, the situation for extremal black holes in the aspects under discussion cannot be considered as a limit $\kappa \to 0$ of that for near-extremal black holes. Actually, we have two distinct situations.

1) Near-extremal black holes. Here, black hole parameters are free ones that change contiuously in some interval. The radius of the circular orbit r_0 near the horizon is not arbitrary but is defined by black hole parameteres from equilibrium conditions [12].

2) Extremal black holes. Here, black hole parameters are fixed but the quantity N_0 that characterizes the location of the orbit is a small free parameter.

In a sense, both cases are complimentary to each other.

To gain further insight, how it happens, let us consider briefly the case of near-extremal black holes (for more details one can consult ref. [12]). Near the horizon,

$$N^2 \approx 2\kappa x + Dx^2, \tag{93}$$

$$\omega \approx \omega_H - b_1 x, \tag{94}$$

where $x = \rho - \rho_H$, κ is a small but nonzero parameter, D and b_1 are constants. Then, eq. (31), (32) give us, in the main approximation (say, for the marginally bound orbit)

$$\sqrt{2\kappa x + Dx^2} b_1 L = (\kappa + Dx)\sqrt{Y_H}. \tag{95}$$

Taking into account (52), (94), (93), we see that $b_1 = B_1\sqrt{D}$, whence

$$B_1 L = \sqrt{Y_H} \frac{\kappa + Dx}{\sqrt{2\kappa Dx + D^2 x^2}}. \tag{96}$$

If $\kappa \neq 0$, it follows from (96) that

$$B_1 L > \sqrt{Y_H}. \tag{97}$$

In view (60), it is seen that for the existence of the circular orbit, (67) should be satisfied. It is worth noting that eq. (97) corresponds to eq. (44) of [12] in which one should require positivity of the quantity L_0^2.

We can make the substitution $x = \kappa D\alpha$ and obtain the equation with respect to α,

$$\sqrt{\alpha^2 + 2\alpha} B_1 L_0 = (1 + \alpha)\sqrt{Y_H}. \tag{98}$$

For the marginally bound orbit, we should insert $L_0 = \frac{m}{\omega_H}$ in (98). Its solution gives us α as a quite definite function of black hole parameters B_1, ω_H, g_H, so the location of the orbit near the horizon is fixed. This is done for any small κ.

In a similar way, for the photon orbit we substitute $m = 0$ into (98). Then, we have another equation for α:

$$\sqrt{\alpha^2 + 2\alpha} B_1 = \frac{1 + \alpha}{\sqrt{g_H}}. \tag{99}$$

By contrast, if $\kappa = 0$ exactly, we obtain from (95) the equation $x(B_1 L - \sqrt{Y_H}) = 0$. As the orbit does not lie on the horizon, $x \neq 0$, and we infer from it that the condition (64) is to be satisfied.

Thus we see that a nontrivial play of small quantities x and κ takes place in such a way that the same eqs. (31), (32) constrain the different entities for near-extremal and extremal black holes. In the first case, it is the location of the circular orbit, in the second one it is the restriction on black hole parameters.

Let us consider now the situation for the ISCO. Then, direct calculations of the second derivative in eq. (71) shows that eqs. (64), (66) are indeed satisfied. (This corresponds to eq. 37 of [12].) Now E and L_0 themselves can be found from these equations and give rise to eq. (66). Eq. (32) for the ISCO requires more terms in the expansion then in (95) and leads to (33), as is shown in [12] in agreement with the previous results on the Kerr metric [5].

For all thee types of orbits, inequality (67) on black hole parameters should be satisfied but, unlike the extremal case, it does not select any discrete values of them. Say, for the Kerr-Newman metric one can only infer from (67) that $\frac{a}{M} > \frac{1}{2}$.

Let us summarize which equations govern the behavior of which orbits. For near-extremal black holes, these are eq. (98) with $E = m$ for the marginally bound orbit or eq. (99) for the photon orbit. For the ISCO, this is eq. (64) and one more equation that is the consequence of (31) and (32). It looks more complicated and can be found in eqs. (36), (39) - (41) of [12]. In all three cases one finds x_0 as the function of the black hole parameters.

For pure extremal black holes the equaitons under discussion are (64) with $E = m$ (the marginally bound orbit) or (64) with $m = 0$ (the photon orbit). For the ISCO these are (64) and eq. (72) (where for the Kerr-Newman metric S should be taken from (80)). The aforementioned equations give constraints on black hole parameters. The radius of the orbit is arbitrary but in the near-horizon region is restricted by the condition $N_0 \ll 1$.

X. VELOCITY ON CIRCULAR NEAR-HORIZON ORBITS

It was observed in [1] that in the extremal limit of the Kerr metric the velocity V measured by a locally nonrotating observer is not equal to 1, as one could naïvely expect. For the ISCO, it was found that $V = \frac{1}{2}$. For the marginally bound orbit, it turned out that $V = \frac{1}{\sqrt{2}}$.

Now, we generalize these results, obtain the similar ones for pure extremal black holes and compare the both.

If a particle moves in the background of the stationary metric (1), the following relation holds:

$$X = \frac{mN}{\sqrt{1 - V^2}}, \tag{100}$$

see eq. (15) of [16]. Taking into account also eq. (58) for the circular orbit, one obtains

$$V = \frac{L_0}{\sqrt{L_0^2 + m^2 g_H}}. \tag{101}$$

A. Near-extremal black holes

Different orbits should be considered separately.

1. Marginally bound orbit

Now, taking into account (69), we obtain

$$V = \frac{1}{\sqrt{1 + \omega_H^2 g_H}}. \tag{102}$$

For the extremal Kerr-Newman black hole, it is seen from (75) and (79) that $\omega_H^2 g_H = \frac{a^2}{M^2}$, so we obtain

$$V = \frac{M}{\sqrt{M^2 + a^2}}. \tag{103}$$

In the extremal Kerr case, $a = M$, so $V = \frac{1}{\sqrt{2}}$ in agreement with [1].

2. ISCO

In this case, taking into account (66), one obtains

$$V = \frac{1}{B_1 \sqrt{g_H}}. \tag{104}$$

It is implied that (67) is satisfied, so $V < 1$.

For the extremal Kerr-Newman metric, it folows from (75) and (77) that $B_1 \sqrt{g_H} = 2\frac{a}{M}$, so

$$V = \frac{M}{2a}, \tag{105}$$

where now $a > \frac{M}{2}$ for the existence of the orbit under discussion. In the extremal Kerr case, $a = M$ and we obtain $V = \frac{1}{2}$ in agreement with [1].

3. Photon circular orbit

Now, by analogy with the massive case, we can introduce $X = \nu_0 - \omega L$, where ν_0 is the conserved quantity having the meaning of the frequency measured at infinity, ν is the locally measured frequency [16]. Instead of (100), now

$$\nu_0 - \omega L = \nu N, \tag{106}$$

(see eq. 40 of [16]). In this case, instead of velocity, it makes sense to speak about the effective gamma-factor $\gamma = \frac{\nu}{\nu_0}$. Using (58) with $m = 0$, one obtains for the near-critical particle with $\nu_0 \approx \omega_H L$ that

$$\gamma = \frac{1}{\omega_H \sqrt{g_H}}. \tag{107}$$

For the Kerr-Newman case, it is seen from (75), (79) that $\omega_H \sqrt{g_H} = \frac{a}{M}$. After substitution into (107), one finds

$$\gamma = \frac{M}{a}. \tag{108}$$

In the extremal Kerr case $\gamma = 1$.

B. Extremal black holes

The previous formulas for the velocity are valid but with the additional constraint that now (64) should be satisfied. Correspondingly, for the Kerr-Newman case a is no longer a free parameter but equal to a_0, where a_0 should be taken from eq. (81) and substituted into (103) or (108). Below, we list the values of the velocity on near-horizon circular orbits for the Ker-Newman metric.

1. Marginally bound orbit

By substitution $a = \frac{1}{\sqrt{3}}$ into (103), we get

$$V = \frac{\sqrt{3}}{2} \tag{109}$$

2. ISCO

Now, $a = \frac{1}{\sqrt{2}}$, and we find from (105) that

$$V = \frac{1}{\sqrt{2}}. \tag{110}$$

Thus, the same value $V = \frac{1}{\sqrt{2}}$ is obtained for the marginally bound orbit in the near-extremal case and for the ISCO in the pure extremal one.

3. Photon orbit

For this orbit, $a = \frac{1}{2}$, so (108) gives us

$$\gamma = 2. \tag{111}$$

Thus we see that the values of the velocity or Lorentz factor on the near-horizon orbit cannot be obtained as a limit $\kappa \to 0$ of these values on orbits in the vicinity of near-extremal black holes.

C.   Radially moving critical particle

For comparison, we also consider the case when a particle is exactly critical, $E = \omega_H L$. The particle's orbit is not circular, the particle approaches the horizon. Then, using (53) instead of (58), one finds from (100) that

$$V = \sqrt{1 - \frac{\omega_H^2 m^2}{B_1^2 E^2}}. \tag{112}$$

For the Kerr-Newman case, $\frac{B_1}{\omega_H} = 2$, so

$$V = \sqrt{1 - \frac{m^2}{4E^2}}. \tag{113}$$

If a particle falls from infinity with the zero initial velocity, $m = E$. Then, $V = \frac{\sqrt{3}}{2}$.

XI. GENERAL FEATURES OF CIRCULAR AND WOULD-BE CIRCULAR ORBITS IN THE NEAR-HORIZON REGION

Thus there are two typical kinds of circular or almost circular orbits in the immediate vicinity of the horizon. (i) There exist true circular orbits that require the particle to be very close to the critical state (74) but not coincide with the critical one nonetheless. For a fixed black hole metric, this is possible for some special values of parameters only. Say, for the Kerr-Newman black hole the parameter $\frac{a}{M} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}$ (plus corrections $O(N_0)$) for the ISCO, marginally bound and photon orbits respectively. (ii) One can specify $X_H = 0$ (or take $X_H = o(N_0^2)$).Then, circular orbits do not exist at all in the near-horizon region. Instead, a particle inspirals approaching the horizon asymptotically that takes an infinitely long proper time [17], [13], [15]. The rate is either exponential or power-like depending on a black hole parameters.

In the case of the extremal Kerr-Newman metric the concrete characteristic values of $\frac{a}{M}$ found above coincide with those in [4] the first approximation. However, interpretation is qualitatively different. In case (i) the true circular orbits of massive particles are absent from the horizon $N = 0$. In case (ii) these values determine the rate with which the particle approaches the horizon.

XII. RELATION TO THE BAÑADOS-SILK-WEST EFFECT AND THE KINEMATIC CENSORSHIP

In the near-horizon region, N is small, so on the circular orbit X is also small according to (43). This is realized for small surface gravity κ (33), (32), i.e. for near-extremal black holes. Meanwhile, such orbits play an essential role in the so-called the Bañados-Silk-West (BSW) effect [8]. Namely, if two particles collide near a black hole, the energy centre of mass $E_{c.m.}$ grows unbound, provided one of particles is critical or near-critical. For the orbits under discussion, according to (58), X has the order N, so a corresponding particle is near-critical. Let us consider in more detail, how the properties of the orbit are related to the BSW effect.

Let two particles with masses m_1 and m_2 collide. By definition, the energy in the centre

of mass

$$E^2_{c.m.} = -P_\mu P^\mu = m_1^2 + m_2^2 + 2m_1 m_2 \gamma, \tag{114}$$

where $P^\mu = p_1^\mu + p_2^\mu$ is their total momentum, $p_{1.2} = m_{1,2} u^\mu_{1,2}$, the Lorentz factor of relative motion

$$\gamma = -u_{1\mu} u_2^\mu. \tag{115}$$

Using the expansion (39), one obtains

$$\gamma = \frac{1}{2} \left(\frac{\beta_1}{\alpha_2} + \frac{\beta_2}{\alpha_1} \right). \tag{116}$$

If a particle 1 orbits a black hole near the horizon while particle 2 is usual, it follows from (46), (47) that

$$\gamma \approx \frac{1}{2N} \frac{\beta_2}{q_1} \tag{117}$$

becomes unbound. The above formulas can be viewed as modification of approach of [7], where it was applied to radial motion, to the case of orbital motion.

On one hand, the proximity of the four-velocity to the horizon generator is due to the fact that the coefficient at l^μ in (39) $\alpha_1^{-1} = O(N^{-1})$ is much larger than the coefficient at N^μ equal to $\beta_1 = O(N)$. From the other hand, the ratio $\frac{\beta_2}{\alpha_1}$ enters the expression for the Lorentz factor, so the same parameter N^{-1} controls both phenomena.

In this context, it is worth reminding that collision between one critical and one usual particle leads to unbound $E_{c.m.} = O(N^{-1/2})$ [15]. Had the collision been possible on the horizon exactly $(N = 0)$ we would have obtained infinite $E_{c.m.}$ And, as a proper time required to reach the horizon is finite for a usual particle, these divergences would have been, at least in principle, observable. This would be unphysical: in any event the energy that can be obtained in any frame cannot be infinite (it can be called "principle of kinematic censorship"). Fortunately, the trajectory of a massive particle on the horizon is impossible, so the experiment under discussion is impossible as well.

However, there is one more potentially dangerous scenario. It is realized when particle 1 is massless that needs a separate treatment since in this case the orbit of particle 1 is not fake. In this case, eqs. (39), (116) are not applicable directly since the particle is characterized by a light-like wave vector k^μ (or $p^\mu = \hbar k^\mu$, where $\hbar$ is the Planck constant) instead of the time-like vector u^μ. Formulas (114) and (115) need some modification. Then,

$$E^2_{c.m.} = m^2 + 2m\gamma, \tag{118}$$

$$\gamma = -u_\mu p^\mu. \tag{119}$$

According to general rules, if a photon 1 is critical and the massive particle 2 is usual, γ grows unbound for collisions near the horizon [16].

Let a usual massive particle 2 cross the extremal horizon where it collides with the photon (particle 1) that follows the horizon generator exactly. Is the quantity $E_{c.m.}$ finite or infinite?

To answer this question, we need to evaluate the scalar product (119). It is conveneint to use the coordinate system in which the metric has the form (16). The terms with $\mu = \bar{t}$ and $\mu = \bar{\phi}$ do not contribute since for the photon under consideration $\bar{E} = 0 = \bar{L}$ as is explained in Sec. VI. Taking also into account that for photon moving along the horizon $\rho = const$, we obtain that

$$- p_\mu u^\mu = (g_\phi \omega^2 - 1)_H \left(\frac{d\rho}{d\tau}\right)_2 \left(\frac{d\bar{t}}{d\lambda}\right)_1, \tag{120}$$

λ is an affine parameter for the photon trajectory, we took into account (16) and (15).

But $\left(\frac{d\rho}{d\tau}\right)_2$ is finite as it follows from (7). It is easy to understand that $\left(\frac{d\bar{t}}{d\lambda}\right)_1$ should be finite as well since this quantity is defined in any point of a horizon for any value of an affine parameter λ and the coordinates (16) are regular on the horizon. Therefore, a particle cannot have infinite $\left(\frac{d\bar{t}}{d\lambda}\right)_1$ in all points of its trajectory. As a result, the Lorentz factor of relative motion γ and $E_{c.m.}$ are also finite. Therefore, the kinematic censorship is preserved.

According to general picture described in detail in [15], [16], collision between one usual and one critical particle leads to the BSW effect. However, in all cases discussed in the aforementioned references, there exists a small but nonzero parameter that controls the process. For a particle slowly moving in a radial direction and approaching the extremal horizon this is τ^{-1} since the proper time τ required for the critical particle to reach the horizon is infinite [17], [13], [15], for collision inside the nonextremal horizon it is proximity of the point of collision to the bifurcation surface [18], for the circular orbit in the background of the near-extremal black hole it is small κ, etc.

Now, we can add one more case to this number of cases. In the situation discussed in the present work, where is no small parameter but another factor prevents infinite $E_{c.m.}$. Either there is no critical trajectory (the massive case) or, as an exception, general rule does not work and collision does not give rise to infinite $E_{c.m.}$ (a photon moving along the horizon generator).

XIII.   CONCLUSION

We have constructed a coordinate system regular on the horizon that generalizes a similar system for the Kerr metric [6]. With its help, it is shown that the circular near-horizon orbits of near-extremal black holes asymptotically approach the generators of the horizon on the slices where the new time $\bar{t} = const$ thus generalizing the observation made in [2].

In the extremal case, general approach to the description of circular equatorial orbits near dirty black holes is suggested. It would seem that there exist circular orbits with $r = r_H$ exactly on the horizon but such orbits are fake for massive particles. Circular orbits are possible for near-critical particles. Their radial distance to the horizon can be made as small as one likes but, nonetheless, their radius does not coincide with that of the horizon. If parameters of a black hole satisfy the exact relation corresponding to a fake orbit on the horizon, this has physical meaning despite the fact the orbit is fake. First, they correspond to the value of the parameters (say, the angular momentum of a black hole $a = a_0$) such that around these values (when $a = a_0 + a_1 N_0 + ...$) there exist circular orbits with small N_0. Second, these distance values manifest itself in dynamics, when a fine-tuned (critical) particle moves on orbits that are not exactly circular, slowly approaching the horizon. When parameters of a black hole (say, its angular momentum) cross a_0, this results in the change of the rate with which a particle approaches the horizon asymptotically.

The velocities measured by a local nonrotating observer are found on the near-horizon orbits. In general, they do not coincide with the values obtained in the extremal limits of near-extremal metrics. It is demonstrated that properties of near-horizon circular orbits near extremal black holes cannot be understood as the extremal limit of corresponding properties of near-extremal black holes.

Connection with the BSW effect is revealed. The fact that on-horizon circular orbits on the horizon are fake preserves the kinematic censorship (the finiteness of energy in collisions) from violation.

It would be of interest to extend the approach and results of the present work to the case of nonequatorial orbits.

Acknowledgments

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